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13. ABSTRACT (Maximum 200 words)

For the advection-diffusion equation, the characteristic-Galerkin formulations are obtained by temporal discretization of the total derivative. These formulations, by construction, are Eulerian-Lagrangian, and therefore can handle time-dependent domains without difficulty. The Galerkin/least-squares space-time formulation, on the other hand, is written over the space-time domain of a problem, and therefore can handle time-dependent domains with no implementational difficulty. The purpose of this paper is to compare these two formulations based on error estimates and numerical performance, in the context of the advection-diffusion equation.

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Characteristic-Galerkin and Galerkin/least-squares space-time formulations for the advection-diffusion equation with time-dependent domains

O. Pironneau

Université Paris 6, Analyse Numérique, T 55-65/5, 4 place Jussieu Paris 75005 France

J. Liou and T. Tezduyar

*Department of Aerospace Engineering and Mechanics, Army High-Performance Computing Research Center and Minnesota Supercomputer Institute, University of Minnesota,
1200 Washington Avenue South, Minneapolis, MN 55415, USA*

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For the advection-diffusion equation, the characteristic-Galerkin formulations are obtained by temporal discretization of the total derivative. These formulations, by construction, are Eulerian-Lagrangian, and therefore can handle time-dependent domains without difficulty. The Galerkin/least-squares space-time formulation, on the other hand, is written over the space-time domain of a problem, and therefore can handle time-dependent domains with no implementational difficulty. The purpose of this paper is to compare these two formulations based on error estimates and numerical performance, in the context of the advection-diffusion equation.

0. Introduction

The advection-diffusion equation arises in several systems in fluid mechanics. It is also a model problem for the Navier-Stokes equations. Time-dependent domains are also frequent, such as in problems involving free surfaces, two liquid interfaces and drifting objects.

Although the advection-diffusion equation is linear and much simpler than the Navier-Stokes equations, it is a challenge to numerical analysts when the diffusion coefficient is small. In the finite element framework, there are several methods to circumvent the difficulty. To give a few examples, we can mention the streamline-upwind/Petrov-Galerkin, Galerkin/least-squares, discontinuous Galerkin, and the Lagrangian-Eulerian schemes based on the characteristic method (see [1] for a review). Time-dependent domains add difficulty to the problem because one has to deal with moving meshes and, depending on the formulation, projections and mesh intersections.

The characteristic methods are derived from the analytical solution of the advection-dissipation equation,

Correspondence to: Dr. O. Pironneau, Université Paris 6, Analyse Numérique, T 55-65/5, 4 place Jussieu Paris 75005 France

$$\phi_t + u\nabla\phi + a\phi = f \quad \text{on } Q = \Omega \times]0, T[,$$

and many schemes, such as those proposed by Chorin [2], Bristeau and Dervieux [3], and the particle methods are based on this.

In 1979, adaptation of these methods to the finite element framework was found concurrently by Benque et al. [4], Douglas and Russell [5] and Pironneau [6]. In simple words, the idea is to discretize the total derivative $D\phi/Dt$ in time instead of ϕ_t . Thus

$$\phi_t + u\nabla\phi - \nu \Delta\phi = f \quad \text{on } Q = \Omega \times]0, T]$$

could be discretized in time by

$$\frac{1}{k} [\phi^{n+1} - \phi^n(x - u^n(x)k)] - \nu \Delta\phi^{n+1} = f^n \quad \text{on } \Omega .$$

This is the basic idea. It can be interpreted as a splitting method (one step of transport + one step of diffusion [7]) because it looks as if one convects ϕ first and then applies the diffusion; but the error analysis shows that there is more to the method than just this. In particular, the method is unconditionally stable and in some cases so accurate that, for example, the rotating hill problem can be solved in only one time step, if desired. As pointed out by Bermejo [8], there are also a number of features that are common to this method and the particle method; it is in fact a particle in cell method in which the cut-off function is the hat function of the finite element method, and for which a projection step is performed every time step.

The Galerkin/least-squares space-time (GLS/ST) formulation for fixed spatial domains was applied by Hughes and co-workers and Johnson and co-workers (see e.g. [9–11]), to a large class of fluid dynamics problems, including compressible flows. Tezduyar et al. [12, 13] applied the GLS/ST formulation to problems with time-dependent domains, and provided several numerical examples from incompressible fluid dynamics, including those involving free surfaces, liquid drops, two-liquid flows, and flows with drifting cylinders.

In the GLS/ST formulation, at each time step, one must solve the problem over a space-time slab. For time dependent domains this space-time slab becomes non-cylindrical.

In this paper, we first describe the characteristic-Galerkin and GLS/ST methods and give proofs of their stability, and partial proofs of the error estimates when the spatial domain is a function of time. Then we evaluate these methods on a test problem involving advection-diffusion of an exponential hill in a rotational flow field.

1. The characteristic-Galerkin method

Consider the convection-dissipation problem. Find ϕ such that

$$\phi_t + u\nabla\phi + a\phi = f \quad \text{in } Q = \{x, t: x \in \Omega(t), t \in]0, T[\} , \quad (1)$$

$$\phi(x, 0) = \phi^0(x) \quad \forall x \in \Omega(0) , \quad (2)$$

$$\begin{aligned} \phi(x, t) = g(x, t) \quad \forall x \in \Gamma^-(t) = \{x: [u(x, t) - v(x, t)] \cdot n(x, t) < 0, x \in \partial\Omega(t)\} \\ \forall t \in]0, T[, \end{aligned} \quad (3)$$

where $u, f, \phi^0, g, \Omega, \Gamma$ are given; n is the outer normal to $\partial\Omega(t)$ and $v(x, t)$ is the velocity of the boundary point $x \in \partial\Omega(t)$ at time t . As usual, t is time and $x \in R^d$.

PROPOSITION 1. Assume $\Omega(t)$ is bounded with Lipschitz boundary $\Gamma(t)$. Let n be the normal. If $\phi^0 \in L^2(\Omega)$, $g \in C^0(\partial Q)$ and $a, u, u_x \in L^\infty(Q)$, $f \in L^2(Q)$, then system (1), (3) has a unique solution in $C^0(\Omega)$.

PROOF. The proof is constructive and will be used also to justify the algorithm. Let $\{X(\tau)\}_{0 \leq \tau \leq t}$ be the solution of

$$\frac{d}{d\tau} X(\tau) = \begin{cases} u(X(\tau), \tau) & \text{if } X(\tau) \in \Omega(\tau), \\ v(X(\tau), \tau) & \text{otherwise,} \end{cases} \quad (4)$$

with the boundary condition

$$X(t) = x. \quad (5)$$

If u is the velocity of a fluid, then X is the trajectory of the particle that passes at position x at time t . Problem (4), (5) has a unique solution because u is Lipschitz. In (4), (5), x, t are parameters so X is written $X(x, t; \tau)$; it is also the characteristic of the hyperbolic equation (1). Now we note that

$$\frac{d}{d\tau} \phi(X(x, t; \tau), \tau)|_{\tau=t} = \phi_{,t}(x, t) + u \nabla \phi(x, t). \quad (6)$$

Thus (1) is rewritten as

$$\phi_{,\tau} + a\phi = f \quad (7)$$

and it can be integrated analytically;

$$\phi(x, t) = \exp \left[- \int_0^t a(X(\tau), \tau) d\tau \right] \left[\lambda + \int_0^t f(X(\sigma), \sigma) \exp \left[\int_0^\sigma a(X(\tau), \tau) d\tau \right] d\sigma \right]. \quad (8)$$

To find λ we use (2) and (3). If $X(x, t; 0) \in \Gamma$, then

$$\lambda = g(X(x, t; 0), \tau(x)), \quad (9)$$

where $\tau(x)$ is the largest time ($< t$) such that $X(x, t; \tau) \in \Gamma(\tau)$.

If $X(x, t; 0) \in \Omega(0)$, then

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$$\lambda = \phi^0(X(x, t; 0)). \quad (9')$$

Let Ω^0 be a bounded open set big enough to contain $\Omega(t)$ at all times. Note that

$$\nabla \cdot (u\phi) = \phi \nabla \cdot u + u \nabla \phi. \quad (10)$$

Thus, we also have an existence theorem for the convection-diffusion equation in divergence form. (Recall that $H(\text{div}, \Omega) = \{u \in L^2(\Omega)^n : \nabla \cdot u \in L^2(\Omega)\}$). We also introduce the notation

$$\Sigma = \{x, t : x \in \Gamma(t), t \in]0, T[\}.$$

Let us assume that u can be extended into Ω^0 such that $u \in L^2(0, T; H(\text{div}, \Omega^0))$.

COROLLARY. *With the hypotheses of Proposition 1 and if $u \in L^2(0, T; H(\text{div}, \Omega^0)) \cap L^\infty(Q)$, problem (1')–(3),*

$$\phi_t + \nabla \cdot (u\phi) = f \quad \text{in } Q = \Omega \times]0, T[, \quad (1')$$

has a unique solution in $C^0(0, T; L^2(\Omega))$.

PROOF. Use (10) and Proposition 1 with $a = \nabla \cdot u$.

We also recall a similar result for the convection-diffusion equation.

PROPOSITION 2. *Let Ω be bounded with boundary Γ Lipschitz. The problem*

$$\phi_t + \nabla \cdot (u\phi) - \nu \Delta \phi = f \quad \text{in } Q, \quad (11)$$

$$\phi(x, 0) = \phi^0(x) \quad \text{in } \Omega(0), \quad (12)$$

$$\phi = g \quad \text{on } \Sigma, \quad (13)$$

has a unique solution in L^2 in t , H^1 in x if

$$u_j, u_{i,j} \in L^\infty(Q), \quad f \in L^\infty(Q), \quad \phi^0 \in L^2(\Omega), \quad g \text{ is } L^2 \text{ in } t, \quad H^{1/2}(\Gamma) \text{ in } x, \\ \nu > 0. \quad (14)$$

PROOF. The proof can be found in Ladyzhenskaya et al. [14] and Lions [15].

Extensions of this result to the time dependent domain can be made by using conformal mappings for example which map $\Omega(t)$ into $\Omega(0)$. The problem then is on a fixed domain but the coefficient ν is now time and space dependent.

1.1. Discretization in time

Now for clarity we assume that $\nabla \cdot u = 0$. By (6),

$$\phi_t + u \nabla \phi = \frac{\partial}{\partial \tau} \phi(X(x, t; \tau), \tau) \Big|_{\tau=t}. \quad (15)$$

Thus by using the fact that $X(x, (n+1)k; (n+1)k) = x$, we can say that

$$(\phi_t + u \nabla \phi)^{n+1} \cong \frac{1}{k} [\phi^{n+1}(x) - \phi^n(X^n(x))], \quad (16)$$

where $X^n(x)$ is an approximation of $X(x, (n+1)k; nk)$.

Let us denote by X_1^n an $O(k^2)$ approximation of $X^n(x)$ and by X_2^n an approximation of order $O(k^3)$ (the differences between the indices of X and the precision order are due to the fact that X^n is an approximation of X on a time interval of order k ; a scheme of $O(k^n)$ gives a precision of $O(k^{n+1})$).

For instance,

$$X_1^n(x) = x - u^n(x)k, \quad (17)$$

$$X_2^n(x) = x - u^{n+1/2} \left(x - u^n(x) \frac{k}{2} \right) k, \quad (18)$$

modified near the boundary without losing accuracy (see below) but so that $X_1^n(\Omega^{n+1}) \subset \Omega^n$. Then we derive a scheme for (11):

$$\frac{1}{k} (\phi^{n+1} - \phi^n \circ X_1^n) - \nu \Delta \phi^{n+1} = f^{n+1}. \quad (19)$$

Note that ϕ^{n+1} and $\phi^n \circ X_1^n$ are both defined on Ω^{n+1} .

REMARK 1. A second order scheme could be

$$\frac{1}{k} (\phi^{n+1} - \phi^n \circ X_2^n) - \frac{\nu}{2} \Delta (\phi^{n+1} + \phi^n) = f^{n+1/2}, \quad (20)$$

but there is a difficulty because ϕ^{n+1} and ϕ^n are not defined on the same domain.

We will not prove convergence in the general case. For simplicity, we will assume that the distance between Ω^n and $X_1^n(\Omega^{n+1})$, $\delta(\Omega^n, X_1^n(\Omega^{n+1}))$ is $O(k^2)$. This is not possible unless the normal velocity of the fluid is equal to the normal velocity of the boundary,

$$u \cdot n = v \cdot n \quad \text{on } \Sigma.$$

LEMMA 1. If u is regular and if $X_1^n(\Omega^{n+1}) \subset \Omega^n$, the schemes (19) and (20) are L^2 -stable and if $\delta(\Omega^n, X_1^n(\Omega^{n+1})) = O(k^2)$, it is convergent $O(k)$.

PROOF. We multiply (19) by ϕ^{n+1} and integrate over Ω^{n+1} . Unless explicitly mentioned all norms and integrals are on Ω^{n+1} :

$$\|\phi^{n+1}\|_\nu^2 \equiv |\phi^{n+1}|^2 + \nu k |\nabla \phi^{n+1}|^2 \leq (|f^{n+1}|k + |\phi^n \circ X_1^n|) |\phi^{n+1}|. \quad (21)$$

Now by a change of variable $y = X_1^n(x)$, we find that

$$\begin{aligned} |\phi^n \circ X_1^n|_0^2 &\equiv \int_{\Omega^{n+1}} \phi^n(X_1^n(x))^2 dx \\ &= \int_{X_1^n(\Omega^{n+1})} \phi^n(y)^2 \det[\nabla X_1^n]^{-1} dy \leq |\phi^n|_{0,\Omega^n}^2 (1 + c|u|_{1,x}^d k). \end{aligned} \quad (22)$$

Therefore ϕ^n satisfies

$$\|\phi^n\|_\nu \leq c[\|f\|_{0,Q} + |\phi^0|_{0,\Omega}]. \quad (23)$$

To obtain the error estimate, we proceed exactly in the same way but on the difference $\epsilon^n = \phi^n - \phi(t^n)$ instead. Indeed

$$\frac{1}{k} (\epsilon^{n+1} - \epsilon^n \circ X_1^n) - \nu \Delta \epsilon^{n+1} = \frac{1}{k} (\phi(t^{n+1}) - \phi(t^n) \circ X_1^n) - \frac{D\phi}{Dt}(t^{n+1}) = O(k),$$

when $u \cdot n = v \cdot n$ on the boundary. Thus, (23) gives

$$\|\epsilon_v^n\| \leq C[\|\epsilon^0\|_{\nu,\Omega^0} + |O(k)|].$$

1.2. Approximation in space

Let us now discretize (19) or (20) with the finite element method. Then we obtain a family of methods for which no additional upwinding is necessary: the schemes are unconditionally stable!

For example with (19), assuming that u has zero divergence, a possible scheme would be

$$\begin{aligned} \int_{\Omega^{n+1}} \phi_h^{n+1} w_h^{n+1} dx + k\nu \int_{\Omega^{n+1}} \nabla \phi_h^{n+1} \nabla w_h^{n+1} dx &= \int_{\Omega^{n+1}} \phi_h^n(X_1^n(x)) w_h^{n+1}(x) dx \\ &+ k \int_{\Omega^{n+1}} f^{n+1} w_h^{n+1} dx \quad \forall w_h^{n+1} \in H_{0h}^{n+1}, \phi_h^{n+1} - g_h \in H_{0h}^{n+1}, \end{aligned} \quad (24)$$

where H_{0h}^{n+1} is a space of continuous polynomial functions on a triangulation of Ω^{n+1} , with zero traces on the boundaries and g_h is a polynomial approximation inside Ω of g .

Notice that (24) is a linear system of the type

$$A\Phi^{n+1} = Ag_h + b, \quad \text{where } A_{i,j} = \int_{\Omega^{n+1}} w^i w^j dx,$$

$$b_j = \int_{\Omega^{n+1}} [kf + \phi_h^n(X_1^n(x))] w^j(x) dx,$$

and where $\{w^i\}$ is a basis of H_{0h}^{n+1} and $\phi_h^{n+1} = \sum \Phi_i^{n+1} w^i$.

REMARK 2. A change of variable can be made in the last integral of (24), $x \rightarrow y = X_1^n(x)$, so (24) is equivalent to

$$\begin{aligned} \int_{\Omega^{n+1}} \phi_h^{n+1} w_h \, dx + k \nu \int_{\Omega^{n+1}} \nabla \phi_h^{n+1} \nabla w_h \, dx &= k \int_{\Omega^{n+1}} f^{n+1} w_h \, dx \\ &+ \int_{X_1^n(\Omega^{n+1})} \phi_h^n(y) w_h((X_1^n)^{-1}(y)) \det \nabla X_1^n \, dy \quad \forall w_h \in H_{0h}^{n+1}, \phi_h^{n+1} \in H_{0h}^{n+1}. \end{aligned} \quad (24')$$

PROPOSITION 3. If $X_1(\Omega^{n+1}) \subset \Omega^n$, $\nabla \cdot u = 0$, $u \cdot n|_F = v \cdot n|_F$, scheme (24) is $L^2(\Omega)$ stable even if $\nu = 0$.

PROOF. Replacing w_h by ϕ_h^{n+1} in (24),

$$\begin{aligned} \|\phi_h^{n+1}\|_\nu^2 &= \int_{\Omega^{n+1}} |\phi_h^{n+1}|^2 \, dx + k \int_{\Omega^{n+1}} \nu \nabla \phi_h^{n+1} \nabla \phi_h^{n+1} \, dx \\ &= k \int_{\Omega^{n+1}} f^{n+1} \phi_h^{n+1} \, dx + \int_{\Omega^{n+1}} \phi_h^n(X_1^n(x)) \phi_h^{n+1}(x) \, dx \\ &\leq (k|f^{n+1}|_0 + |\phi_h^n \circ X_1^n|_0) |\phi_h^{n+1}|_0 \\ &\leq (|\phi_h^n|_0(1 + c|u|_{1,x}^d k) + k|f^{n+1}|_0) |\phi_h^{n+1}|_0. \end{aligned} \quad (25)$$

The last inequality is a consequence of the fact that the mapping $x \rightarrow X$ is volume preserving. Finally,

$$|\phi_h^n|_{0,\Omega} \leq (1 + c|u|_{1,x} k)^n \left(\|\phi_h^0\|_{\nu,\Omega} + \sum k |f^n|_{0,\Omega^n} \right). \quad (26)$$

PROPOSITION 4. If H_{0h}^{n+1} is the space of continuous piecewise affine functions on the triangulation of Ω^{n+1} , then the $L^2(\Omega)$ error between ϕ_h^n solution of (24) and ϕ^n solution of (11) is $O(h^2/k + k + h)$. Hence the scheme is $O(h^2/k + k + h)$.

PROOF. Let us subtract (11) from (24) to obtain an error projected on L^2 :

$$\epsilon_h^{n+1} = \phi_h^{n+1} - \Pi_h \phi^{n+1}, \quad (27)$$

where $\Pi_h \phi^{n+1}$ is an interpolation in H_{0h}^{n+1} of ϕ^{n+1} . We obtain

$$\begin{aligned} \int_{\Omega^{n+1}} \epsilon_h^{n+1} w_h \, dx + k \nu \int_{\Omega^{n+1}} \nabla \epsilon_h^{n+1} \nabla w_h \, dx &- \int_{\Omega^{n+1}} \epsilon_h^n \circ X_1^n w_h \, dx \\ &= \int_{\Omega^{n+1}} (\phi^{n+1} - \Pi_h \phi^{n+1}) w_h \, dx + \nu k \int_{\Omega^{n+1}} \nabla (\phi^{n+1} - \Pi_h \phi^{n+1}) \nabla w_h \, dx \\ &- \int_{\Omega^{n+1}} (\phi^n - \Pi_h \phi^n) \circ X_1^n w_h \, dx. \end{aligned} \quad (28)$$

From (28), with $w_h = \epsilon_h^{n+1}$, we find

$$\|\epsilon_h^{n+1}\|_\nu^2 \leq ((1 + c|u|_{1,x}^d k) \|\epsilon_h^n\|_\nu + \|\phi^{n+1} - \Pi_h \phi^{n+1}\|_\nu + (1 + c|u|_{1,x}^d k) \|\phi^n - \Pi_h \phi^n\|_0) \|\epsilon_h^{n+1}\|_\nu,$$

so

$$\|\epsilon_h^{n+1}\|_\nu \leq (1 + c|u|_{1,x}^d k) \|\epsilon_h^n\|_\nu + C(h^2 + \nu kh). \quad (29)$$

REMARK 4. By comparing with (24), we see that ϵ_h is solution of a problem similar to that of ϕ_h in which f would be replaced by

$$\frac{1}{k} (\phi^{n+1} - \Pi_h \phi^{n+1}) - \nu \Delta_h (\phi^{n+1} - \Pi_h \phi^{n+1}) - \frac{1}{k} (\phi^n - \Pi_h \phi^n) \circ X_1^n,$$

where Δ_h is an approximation of Δ . We have bounded the first and the last terms independently. By being more cautious one can show that the error is in fact $O(h + k + \min(h^2/k, h))$ [5] but the dependence on ν is stronger.

The case $\nu = 0$: Then (24) becomes

$$\begin{aligned} \int_{\Omega^{n+1}} \phi_h^{n+1} w_h \, dx &= \int_{\Omega^{n+1}} \phi_h^n \circ X_1^n w_h \, dx + k \int_{\Omega^{n+1}} f^{n+1} w_h \, dx \\ \forall w_h \in H_{0h}^{n+1}, \quad \phi_h^{n+1} &\in H_{0h}^{n+1}; \end{aligned} \quad (30)$$

that is,

$$\phi_h^{n+1} = \Pi_h (\phi_h^n \circ X_1^n + k f^{n+1}), \quad (31)$$

where Π_h is the projection operator from L^2 into H_{0h}^{n+1} . From this equation we see that the fewer time steps there are the better because a projection inevitably produces numerical diffusion; however when k is large the computation of X^n is costly. Experience shows that $k \approx 1.5h/u$ is a good choice. This is a striking behavior of these Lagrangian/Eulerian methods: they should be used with a $\text{CFL} > 1$.

Conservativity: Assume that

$$u \cdot n = v \cdot n \quad \text{on } \Sigma, \quad (32)$$

and that $f = 0$ and $\nu = 0$; the equation is

$$\phi_t + u \nabla \phi = 0, \quad \phi(x, 0) = \phi^0(x). \quad (33)$$

By hypothesis $\nabla \cdot u = 0$, so by integrating (33),

$$\int_{\Omega(t)} \phi(t, x) \, dx = \int_{\Omega(0)} \phi^0(x) \, dx \quad \forall t. \quad (34)$$

On the other hand, (24') with $w_h = 1$ also gives (34) because $\det|\nabla X^n| = 1$ and $X^n(\Omega^{n+1}) =$

Ω^n . So the scheme is conservative up to quadrature errors on the approximation of the last integral in (24) or (24') and up to integration errors due to X^n being replaced by X_1^n or X_2^n . Indeed $w_h = 1$ in (24) gives

$$\int_{\Omega^{n+1}} \phi_h^{n+1}(x) dx = \int_{\Omega^{n+1}} \phi_h^n \circ X_1^n dx = \int_{X_1^n(\Omega^{n+1})} \phi_h^n(y) \det|\nabla X_1^n|^{-1} dy. \quad (35)$$

Thus, if $\det|\nabla X_1^n| = 1$ (which implies $X_1^n(\Omega^{n+1}) = \Omega^n$), we obtain

$$\int_{\Omega^{n+1}} \phi_h^{n+1}(x) dx = \int_{\Omega^n} \phi_h^n(y) dy = \int_{\Omega^0} \phi^0(x) dx. \quad (36)$$

Thus, the scheme is conservative up to numerical integration errors on X_1^n . In many applications, this may not be sufficient but (17) or (18) can be modified so that $\det|\nabla X_1^n| = 1$. Indeed since $\nabla \cdot u = 0$, there exists a stream function (vector in 3D) ψ such that $u = \nabla \times \psi$. Let ψ_h be a continuous piecewise approximation of ψ . Then the ODE $dX/d\tau = \nabla \times \psi_h$ can be integrated exactly ($\{X(\tau)\}$ is a broken line) and $x \rightarrow X_1^n(x)$ is volume preserving.

1.3. Implementation problems

Two points remain to be clarified: (i) the computation of $X^n(x)$ and (ii) the computation of the integral,

$$I^n = \int_{\Omega} \phi_h \circ X_h^n w_h dx. \quad (37)$$

1.3.1. Computation of (37)

Primal Gauss formula. We use a Gauss quadrature formula,

$$I^n \cong \sum \omega^k \phi_h(X_h^n(\xi^k)) w_h(\xi^k); \quad (38)$$

for instance with P^1 elements we can take:

- (a) $\{\xi^k\}$ = all the mid-points of the edges and $\omega^k = \sigma_k/3$ in 2D, $\sigma_k/4$ in 3D, where σ_k is the area (volume) of the elements which contain ξ^k ;
- (b) the 3 (4 in 3D) inner nodes Gauss formula [16, 17];
- (c) the 7 integration points formula (vertices, mid-edges and central node).

Any other integration formula can be used as long as the weights are positive. Numerical tests can be found below with triangles and in [18] with quadrangles and the 9 nodes formula.

Dual Gauss formula. Denote by $X_h^{n+}(y)$ the inverse mapping of $x \rightarrow y = X_h^n(x)$. Obviously $X_h^{n+}(y)$ is an approximation of the solution at $(n+1)k$ of $dX/d\tau = u(X(\tau), \tau)$, $X(nk) = y$. Another class of methods can be obtained if a change of variable is made in the integral as in (24').

$$\int_{\Omega} \phi_h \circ X_h^n w_h dx = \int_{X_h^n(\Omega)} \phi_h(y) w_h(X_h^{n+}(y)) \det|\nabla X_h^{n+}| dy. \quad (39)$$

When $\nabla \cdot u = 0$, the term $\det|\nabla X_h|$ is 1, so a Gauss formula gives

$$I \cong \sum \omega^k \phi_h(\xi^k) w_h(X^{n+1}(\xi^k)). \quad (40)$$

Now with such a quadrature formula, the scheme is conservative regardless of quadrature errors. Such schemes were introduced and thoroughly tested by Benque et al. [4]. However, the effects of quadrature errors on stability and error estimates is an open problem.

1.3.2. Computation of $X^n(x)$

To compute $\phi_h^n(X^n(\xi^k))$ one must answer the following: given ξ^k find l such that $X^n(\xi^k) \in T_l$, a triangle of the triangulation of Ω^n . This is not a simple problem. We shall proceed in two steps.

Step 1. Find m such that $\xi^k \in T_m$, a triangle of the triangulation of Ω^n . In our problem we know only in which triangle of Ω^{n+1} ξ^k lies; we do not know that information in the triangulation of Ω^n . So assume we know $\{m', \xi'^k\}$ with ξ'^k near ξ^k and $\xi'^k \in T_{m'}$, a triangle of the triangulation of Ω^n . Then intersect the segment $\{\xi'^k, \xi^k\}$ with the triangulation of Ω^n and proceed from neighbor triangle to neighbor triangle, starting from ξ'^k until ξ^k is reached. This is possible if we have an array giving the number of the three neighbor triangles of each triangle of the triangulation of Ω^n .

Step 2. Compute all intersections of $\{\xi'^k, X^n(\xi^k)\}$ with the edges of the triangulation and proceed from neighbor to neighbor until $X^n(\xi^k)$ is reached to find l .

But then at no extra cost we can improve on the scheme to compute $X^n(\xi^k)$: the numerical scheme of $X^n(\xi^k)$ is applied within an element and when another element is reached, the value of u_h on the new element is used. With the first order scheme then, one can use the barycentric coordinates $\{\lambda_i\}$ of ξ^k and proceed as shown in Fig. 1. First on each element compute μ_i such that

$$u(Q) = \sum_{i=1, \dots, d+1} \mu_i q^i \quad \sum \mu_i = 0,$$

where $\{q^i\}$ are the vertices of the element, Q its barycenter and d the dimension of the space.

Then find ρ such that

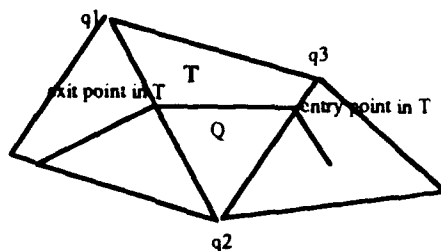


Fig. 1. Computation of X_h^n by following the streamline in the triangulation.

$$\lambda'_i = \lambda_i + \rho \mu_i \Rightarrow \prod_i \lambda'_i = 0, \quad \lambda'_i \geq 0. \quad (42)$$

To find ρ , we assume that m is the index of the λ_m , which is zero:

$$\rho = -\frac{\lambda_m}{\mu_m}, \quad (43)$$

then if $\lambda_i \geq 0, \forall i$, the m is the right one.

Thus most of the work goes into computing μ_i for all the elements. Because of round off errors, it may be difficult at times to decide which is the next triangle that the characteristic will cross, for instance if $X^n(\xi^k)$ is on a vertex; so careful programming is needed.

If $u_h = \nabla \times \psi_h$ and ψ_h is piecewise affine and continuous, as mentioned above, a slightly different algorithm can be used in place of (17). Instead of defining the characteristics as the curves tangent to u_h , we define them as curves of equal values of ψ_h ; the numerical advantage is that if a characteristic enters a triangle it must leave it even in presence of round off errors, i.e. given an entry point ξ^k and $\psi_h(\xi^k)$ one needs just to test which side $q'q''$ is, such that $\psi_h(q^k) \in [\psi_h(q'u), \psi_h(q'')] \text{ or } [\psi_h(q'), \psi_h(q'')] \text{ and then obtain the exit point by linear interpolation based on } \psi_h$.

2. Galerkin least-squares space-time formulation

Consider again the convection-diffusion equation

$$\phi_t + \nabla \cdot (u\phi) - \nu \Delta \phi = f \quad \text{on } Q, \quad (44)$$

$$\phi(x, 0) = \phi^0(x) \quad \text{in } \Omega(0), \quad (45)$$

$$\phi = g \quad \text{on } \Sigma. \quad (46)$$

Assume for simplicity that $\nabla \cdot u = 0$.

Choose a time step k and construct a quadrangulation T_h^n of $\Omega(t^n)$, $t^n = nk$, for each n such that the quadrangulation T_h^{n+1} of $\Omega(t^{n+1})$ is obtained by moving the vertices of that of $\Omega(t^n)$ with the condition that no quadrangle is flipped over in the process. This way, by joining the vertices of T_h^n with their images in T_h^{n+1} , one obtains a proper quadrangulation of the space time slab

$$Q^{n+1} = \{(x, t): x \in \Omega(t), t \in]t^n, t^{n+1}[\}. \quad (47)$$

REMARK. The method does not require that the mesh at the next time step should be obtained by moving the vertices of the previous mesh, but our proof of convergence relies on this special case.

Define the space of piecewise linear functions which are piecewise linear in *each* component of the vector (x, t) of R^{d+1} , which are continuous in x but discontinuous in t :

$$\begin{aligned}
W_h^{n+1} &= \{w_h \in C^0(Q^{n+1}): w_h \text{ piecewise affine in } x_i, \text{ affine in } t \text{ on } Q_h^n\}, \\
W_{0h}^{n+1} &= \left\{w_h \in W_h^{n+1}: w_h = 0 \text{ on } \Sigma\right\}.
\end{aligned} \tag{48}$$

The Galerkin least-squares space-time (GLS/ST) method defines the approximate solution ϕ_h^{n+1} as the solution in W_h^{n+1} , equal to g_h on Σ , and such that for all w_h in W_{0h}^{n+1} we have

$$\begin{aligned}
&\int_{Q^{n+1}} (\phi_{h,t}^{n+1} + u \cdot \nabla \phi_h^{n+1} - f) w_h + \int_{Q^{n+1}} \nu \nabla \phi_h^{n+1} \cdot \nabla w_h \\
&\quad + \sum_{e=1}^{n_{el}} \int_{(Q^{n+1})^e} \tau (\phi_{h,t}^{n+1} + u \cdot \nabla \phi_h^{n+1} - \nu \Delta \phi_h^{n+1} - f), \\
&(w_{h,t} + u \cdot \nabla w_h - \nu \Delta w_h) + \int_{\Omega^n} (\phi_h^{n+1} - \phi_h^n) w_h = 0,
\end{aligned}$$

where n_{el} is the number of elements. For the purpose of proving well-posedness and convergence, we use the following modified form of (49):

$$\begin{aligned}
&\int_{Q^{n+1}} [\phi_{h,t}^{n+1} + u \nabla \phi_h^{n+1} - f][w_h + \tau(w_{h,t} + u \nabla w_h)] + \int_{\Omega^n} (\phi_h^{n+1} - \phi_h^n) w_h \\
&\quad + \nu \int_{Q^{n+1}} \nabla \phi_h^{n+1} \nabla w_h = 0.
\end{aligned} \tag{50}$$

The spatial domain Ω^n is to be understood as $\Omega^n \times \{t^n\}$. The scalar τ is a positive parameter of order $O(h+k)$.

PROPOSITION 5. *Problem (50) is well-posed; the solution exists and is unique.*

PROOF. Equation (50) is a linear system with respect to the values of ϕ_h^{n+1} at the vertices of the quadrangulation of Q^{n+1} , i.e. the vertices of both T_h^n and T_h^{n+1} . There are obviously as many equations as unknowns so we only need to check that the kernel of the system is zero, i.e., that $f = g = \phi^0 = 0$ implies $\phi_h^{n+1} = 0$ for all n .

With these zero data and $w_h = \phi_h^{n+1}$, (50) is

$$\begin{aligned}
&\int_{Q^{n+1}} \left[\frac{1}{2} (\phi_{h,t}^{n+1})^2 + \frac{1}{2} u \nabla (\phi_h^{n+1})^2 \right] + \int_{\Omega^n} [(\phi_h^{n+1})^2 - \phi_h^n \phi_h^{n+1}] \\
&\quad + \nu \int_{Q^{n+1}} |\nabla \phi_h^{n+1}|^2 = 0 + \int_{Q^{n+1}} \tau (\phi_{h,t}^{n+1} + u \cdot \nabla \phi_h^{n+1})^2.
\end{aligned} \tag{51}$$

Let us integrate the first two terms by parts:

$$\begin{aligned}
&\int_{Q^{n+1}} [(\phi_{h,t}^{n+1})^2 + u \nabla (\phi_h^{n+1})^2] = \int_{\Omega^{n+1}} (\phi_h^{n+1})^2 - \int_{\Omega^n} (\phi_h^{n+1})^2 \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{\partial \Omega(t)} (\phi_h^{n+1})^2 (u - v) \cdot n \, dx \, dt.
\end{aligned} \tag{52}$$

Now the last integral is positive because ϕ_h^{n+1} is zero when $u \cdot n < v \cdot n$. Therefore (50) yields

$$\frac{1}{2} \int_{\Omega^{n+1}} (\phi_h^{n+1})^2 - \frac{1}{2} \int_{\Omega^n} (\phi_h^{n+1})^2 + \int_{\Omega^n} [(\phi_h^{n+1})^2 - \phi_h^n \phi_h^{n+1}] \leq 0. \quad (53)$$

Define

$$a_n = \left(\int_{\Omega^n} (\phi_h^n)^2 \right)^{1/2}, \quad b_{n+1} = \left(\int_{\Omega^n} (\phi_h^{n+1})^2 \right)^{1/2}. \quad (54)$$

Then, using Schwartz' inequality, (53) is also

$$a_{n+1}^2 + b_{n+1}^2 \leq 2a_n b_{n+1}. \quad (55)$$

But this can be rewritten as

$$a_{n+1}^2 - a_n^2 + (b_{n+1} - a_n)^2 \leq 0; \quad (56)$$

therefore

$$a_{n+1} \leq a_n. \quad (57)$$

Now $a_0 = 0$ by hypothesis and $a_n \geq 0$, so (56) implies $a_n = 0$.

Let us prove convergence. For simplicity we will do it in the hyperbolic case only. When $\nu \neq 0$, the method converges $O(h + k)$ in H^1 . Extension to the general case can be established following [11].

PROPOSITION 6. When $\tau = O(h + k)$, $\nu = 0$, $g = g_h$, the method converges $O((k + h)^{3/2})$ in L^2 .

PROOF. The proof is an adaptation of that given by Johnson [11]. The idea is not to treat space and time separately. Let us introduce the vector

$$U = (1, u)^t \in R^{d+1} \quad (58)$$

and rewrite (44), for $\nu = 0$, as

$$U \nabla_{x,t} \phi = f. \quad (59)$$

Thus (50) is equivalent to

$$\int_{Q^{n+1}} [U \nabla_{x,t} \phi_h^{n+1} - f][w_h + \tau U \nabla_{x,t} w_h] + \int_{\Omega^n} [\phi_h^{n+1} - \phi_h^n] w_h = 0. \quad (60)$$

The only difference with the time independent formulation of the least-square Galerkin method is the fact that the elements are discontinuous in time and the presence of the last integral in (60) which looks like a penalty term to insure that ϕ_h^{n+1} is close to ϕ_h^n at $t = t^n$.

Therefore, let Π_h be an interpolation operator from $H^1(Q)$ into ΠW_h^n and define

$$\psi_h = \Pi_h \phi, \quad \epsilon = \phi - \phi_h, \quad \epsilon_h = \psi_h - \phi_h. \quad (61)$$

Subtract (59) in variational form from (60):

$$\int_{Q^{n+1}} U \nabla_{x,t} \epsilon^{n+1} [w_h + \tau U \nabla_{x,t} w_h] + \int_{\Omega^n} [\epsilon^{n+1} - \epsilon^n] w_h = 0. \quad (62)$$

Notice that

$$\epsilon = \epsilon_h - (\psi_h - \phi). \quad (63)$$

Choose $w_h = \epsilon_h^{n+1}$ and rewrite (62) as

$$\begin{aligned} & \int_{Q^{n+1}} (U \nabla_{x,t} \epsilon_h^{n+1}) [\epsilon_h^{n+1} + \tau U \nabla_{x,t} \epsilon_h^{n+1}] + \int_{\Omega^n} [\epsilon_h^{n+1} - \epsilon_h^n] \epsilon_h^{n+1} \\ &= \int_{Q^{n+1}} U \nabla_{x,t} (\psi_h^{n+1} - \phi^{n+1}) [\epsilon_h^{n+1} + \tau U \nabla_{x,t} \epsilon_h^{n+1}] \\ &+ \int_{\Omega^n} [(\psi_h^{n+1} - \phi^{n+1}) - (\psi_h^n - \phi^n)] \epsilon_h^{n+1}. \end{aligned} \quad (64)$$

Next notice that if N denotes the space-time normal to ∂Q , we have

$$\int_{Q^{n+1}} (U \nabla_{x,t} \epsilon_h^{n+1}) \epsilon_h^{n+1} = \frac{1}{2} \int_{Q^{n+1}} U \nabla_{x,t} (\epsilon_h^{n+1})^2 = \frac{1}{2} \int_{\partial Q^{n+1}} (\epsilon_h^{n+1})^2 U \cdot N \quad (65)$$

$$= \frac{1}{2} \int_{\Omega^{n+1}} (\epsilon_h^{n+1})^2 - \frac{1}{2} \int_{\Omega^n} (\epsilon_h^{n+1})^2 + \frac{1}{2} \int_{t^n}^{t^{n+1}} \int_{\partial \Omega(t)} (\epsilon_h^{n+1})^2 (u - v) \cdot n. \quad (66)$$

The last integral is positive because ϵ_h is zero when $u \cdot n < v \cdot n$. Using this in (63) and integrating by parts the term $U \nabla_{x,t} (\psi_h^{n+1} - \phi^{n+1}) \epsilon_h^{n+1}$, yields

$$\begin{aligned} & \frac{1}{2} \int_{\Omega^{n+1}} (\epsilon_h^{n+1})^2 + \frac{1}{2} \int_{\Omega^n} (\epsilon_h^{n+1})^2 - \int_{\Omega^n} \epsilon_h^n \epsilon_h^{n+1} + \frac{1}{2} \int_{t^n}^{t^{n+1}} \int_{\partial \Omega(t)} (\epsilon_h^{n+1})^2 (u - v) \cdot n \\ &+ \int_{Q^{n+1}} \tau (U \nabla_{x,t} \epsilon_h^{n+1})^2 \\ &\leq - \int_{Q^{n+1}} [(\psi_h^{n+1} - \phi^{n+1}) U \nabla_{x,t} \epsilon_h^{n+1}] + \int_{\partial Q^{n+1}} (\psi_h^{n+1} - \phi^{n+1}) \epsilon_h^{n+1} U \cdot N \\ &+ \tau |U \nabla_{x,t} (\psi_h^{n+1} - \phi^{n+1})|_{0, Q^{n+1}} |U \nabla_{x,t} \epsilon_h^{n+1}|_{0, Q^{n+1}} \\ &+ \int_{\Omega^n} [\psi_h^{n+1} - \phi^{n+1} - (\psi_h^n - \phi^n)] \epsilon_h^{n+1}. \end{aligned} \quad (67)$$

Finally we rewrite the first three terms as

$$\int_{\Omega^{n+1}} (\epsilon_h^{n+1})^2 + \int_{\Omega^n} (\epsilon_h^{n+1})^2 - 2 \int_{\Omega^n} \epsilon_h^n \epsilon_h^{n+1} = |\epsilon_h^{n+1}|_{0,\Omega^{n+1}}^2 - |\epsilon_h^n|_{0,\Omega^n}^2 + \int_{\Omega^n} (\epsilon_h^n - \epsilon_h^{n+1})^2 \quad (68)$$

and we obtain

$$\begin{aligned} & \frac{1}{2} |\epsilon_h^{n+1}|_{0,\Omega^{n+1}}^2 - \frac{1}{2} |\epsilon_h^n|_{0,\Omega^n}^2 + \frac{1}{2} \int_{\Omega^n} (\epsilon_h^n - \epsilon_h^{n+1})^2 + \frac{1}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\Omega(t)} (\epsilon_h^{n+1})^2 (u-v) \cdot n \\ & + |\tau(U\nabla_{x,t} \epsilon_h^{n+1})|_{0,Q^{n+1}}^2 \\ & \leq - \int_{Q^{n+1}} (\psi_h^{n+1} - \phi^{n+1}) U\nabla_{x,t} \epsilon_h^{n+1} + \int_{\Omega^n} (\psi_h^n - \phi^n) (\epsilon_h^n - \epsilon_h^{n+1}) \\ & + \int_{\Omega^{n+1}} (\psi_h^{n+1} - \phi^{n+1}) \epsilon_h^{n+1} - \int_{\Omega^n} (\psi_h^n - \phi^n) \epsilon_h^n \\ & + \int_{t^n}^{t^{n+1}} \int_{\partial\Omega(t)} (\psi_h^{n+1} - \phi^{n+1}) \epsilon_h^{n+1} (u-v) \cdot n \\ & + c\tau |U\nabla_{x,t} (\psi_h^{n+1} - \phi^{n+1})|_{0,Q^{n+1}} |U\nabla_{x,t} \epsilon_h^{n+1}|_{0,Q^{n+1}} \quad (69) \end{aligned}$$

Here we have used the fact that $\psi^{n+1} = \psi^n$ and $\phi^{n+1} = \phi^n$ on Ω^n (see (61)). It would not be true if the triangulation at step $n+1$ was not obtained by moving the nodes of that at step n .

Now denote $\Sigma_m = \{(x, t): x \in \partial\Omega(t), t \leq t^m\}$ and $Q_m = \{(x, t) \in Q: t \leq t^m\}$. Let us sum with respect to n :

$$\begin{aligned} & |\epsilon_h^m|_{0,\Omega^m}^2 + \sum_{n=0}^{m-1} |\epsilon_h^n - \epsilon_h^{n+1}|_{0,\Omega^n}^2 + \int_{\Sigma_m} \epsilon_h^2 (u-v) \cdot n + 2\tau |U\nabla_{x,t} \epsilon_h|_{0,Q_m}^2 \\ & \leq |\epsilon_h^0|_{0,\Omega^0}^2 + 2|\psi_h - \phi|_{0,Q_m} |U\nabla_{x,t} \epsilon_h|_{0,Q_m} + 2 \int_{\Omega^m} (\psi_h^m - \phi^m) \epsilon_h^m \\ & - 2 \int_{\Omega^0} (\psi_h^0 - \phi^0) \epsilon_h^0 + 2 \sum_n |\psi_h^n - \phi^n|_{0,\Omega^n} |\epsilon_h^n - \epsilon_h^{n+1}|_{0,\Omega^n} \\ & + 2 \int_{\Sigma_m} (\psi_h - \phi) \epsilon_h (u-v) \cdot n + 2\tau |U\nabla_{x,t} (\psi_h - \phi)|_{0,Q_m} |U\nabla_{x,t} \epsilon_h|_{0,Q_m} \quad (70) \end{aligned}$$

Now, since ψ_h is an interpolate of ϕ , all terms on the right are bounded by some power of $\delta = h + k$, the biggest of which is $\delta^{3/2}$ because

$$|\psi_h - \phi|_{0,Q_m} \leq c \|\phi\|_{2,Q} \delta^2, \quad |\nabla_{x,t} (\psi_h - \phi)|_{0,Q_m} \leq c \|\phi\|_{2,Q} \delta, \quad (71)$$

$$|\psi_h - \phi|_{0,S} \leq c \|\phi\|_{2,Q} \delta^{3/2} \quad (72)$$

on any smooth surface S of Q . So the right-hand side of (69) is bounded by

$$c \|\phi\|_{2,Q} \delta \left[(\delta + \tau) |U \nabla_{x,t} \epsilon_h|_{0,Q} + \delta |\epsilon_h^m|_{0,\Omega^m} + \delta \left(\sum_n |\epsilon_h^n - \epsilon_h^{n+1}|_{0,\Omega^n}^2 \right)^{1/2} + \delta^{1/2} |(u-v) \cdot n \epsilon_h^m|_{0,\Sigma} \right]. \quad (73)$$

Assume $\tau = O(\delta)$. Since $\epsilon_h = 0$ when $(u-v) \cdot n < 0$, we have

$$\left[|\epsilon^m|_{0,\Omega^m}^2 + \sum_{n=0}^{n=m-1} |\epsilon^n - \epsilon^{n+1}|_{0,\Omega^n}^2 + \int_{\Sigma_m} \epsilon^2 (u-v) \cdot n + \tau |U \nabla_{x,t} \epsilon|_{0,Q_m}^2 \right]^{1/2} \leq c \|\phi\|_{2,Q} \delta^{3/2}. \quad (74)$$

REMARK. If $\|\phi\|_2$ is replaced by $\|\phi\|_3$, then $\delta^{3/2}$ can be replaced by δ^2 in (72). With this additional regularity and our hypothesis the method is $O(\delta^2)$.

3. Numerical simulations

3.1. Summary of the algorithms

The characteristic-Galerkin method used is based on a finite element method of order 1 (continuous piecewise affine) on triangles with a primal Gauss quadrature with seven integration points, the vertices, the middle of the edges of the triangles and the center of the triangles. Finally the characteristics are computed exactly via a piecewise linear stream function.

Since the domain is time dependent, the triangulations at each time step are different. Here we assume that each triangulation is obtained from the previous time step by moving the vertices with the velocity field v . We have two velocity fields, the mesh velocity v and the fluid velocity u . At each time step the following must be done:

ALGORITHM 1: Characteristic-Galerkin with time dependent domains

- (1) Compute the coordinates of the vertices of the new mesh.
- (2) Compute the old mesh triangle numbers which contain the new mesh quadrature points.
- (3) Find where these quadrature points were in the fluid at the previous time step and their triangle number in the old mesh.
- (4) Compute the integrals just on the right of the equal sign in (24) with $w_h = w^j$ (new mesh); use the seven Gauss point formula.
- (5) Compute the matrix of the linear system (w^i, w^j are with respect to the new mesh)

$$A_{ij} = \int_{\Omega^{n+1}} w^i w^j + \nu \int_{\Omega^{n+1}} \nabla w^i \nabla w^j. \quad (75)$$

- (6) Solve the linear system (by the Choleski algorithm for instance).
- (7) Update all arrays and go back to 1 until $t = T$ is reached.

It should be noted that when $\nu = 0$ and when the motion of the mesh is incompressible ($\nabla \cdot v = 0$), the matrix A does not depend on n . This allows us to program the method to have always only one mesh at a time in the memory. It also allows a unique factorization of A but we have preferred to reconstruct A at every time step.

REMARK. Algorithm 1 computes the solution of (11) in the frame of reference in which the domain is fixed so that the velocity of the mesh should be changed accordingly if the results are desired in the original frame of reference.

The least-squares Galerkin method is conceptually simpler and much easier to program, but it requires the solution of a linear system which is double the size of the other method and the system is non-symmetric. On the other hand it is exactly conservative (the other method is conservative only if dual quadratures are used).

ALGORITHM 2: Least-squares Galerkin

- (1) Triangulate Q^n .
- (2) Construct the linear system of matrix

$$A_{i,j} = \int_{Q^{n+1}} [(w'_{,i} + u \nabla w')(w'_{,j} + \tau(w'_{,i}) + u \nabla w') + \nu \nabla w' \nabla w'] + \int_{\Omega^n} w' w' . \quad (76)$$

- (3) Construct the right-hand side and solve the system.

Contrary to Algorithm 1, the results of Algorithm 2 correspond to the solution of the problem in the fixed frame of reference where the mesh moves.

3.2. The rotating hill

The flow is purely rotational and the convected variable has an exponential hill profile, i.e., almost zero everywhere except in a small region. After one turn, the solution should be identical to the initial condition, if $\nu = 0$.

The velocity of convection is

$$u = \omega \times x = (\omega x_2, -\omega x_1)^t, \quad \omega = 1.$$

The initial condition is

$$\phi^0(x) = \exp(-10|(x - x_0)|^2), \quad x_0 = (0.5 \ 0.0)^t.$$

The domain Ω is the unit circle. The time step chosen is $\delta t = \pi/14$.

Experiments are done with a moving domain and the mesh moves with the velocity of the domain v which is different from the velocity of the fluid u .

$$v = (v_1, v_2)^t, \quad v = \beta(x, t) \times x,$$

$$\beta(x, t) = \beta_0 + \cos(2\pi t)\beta_r \sin(2\pi \alpha r), \quad r = |x|.$$

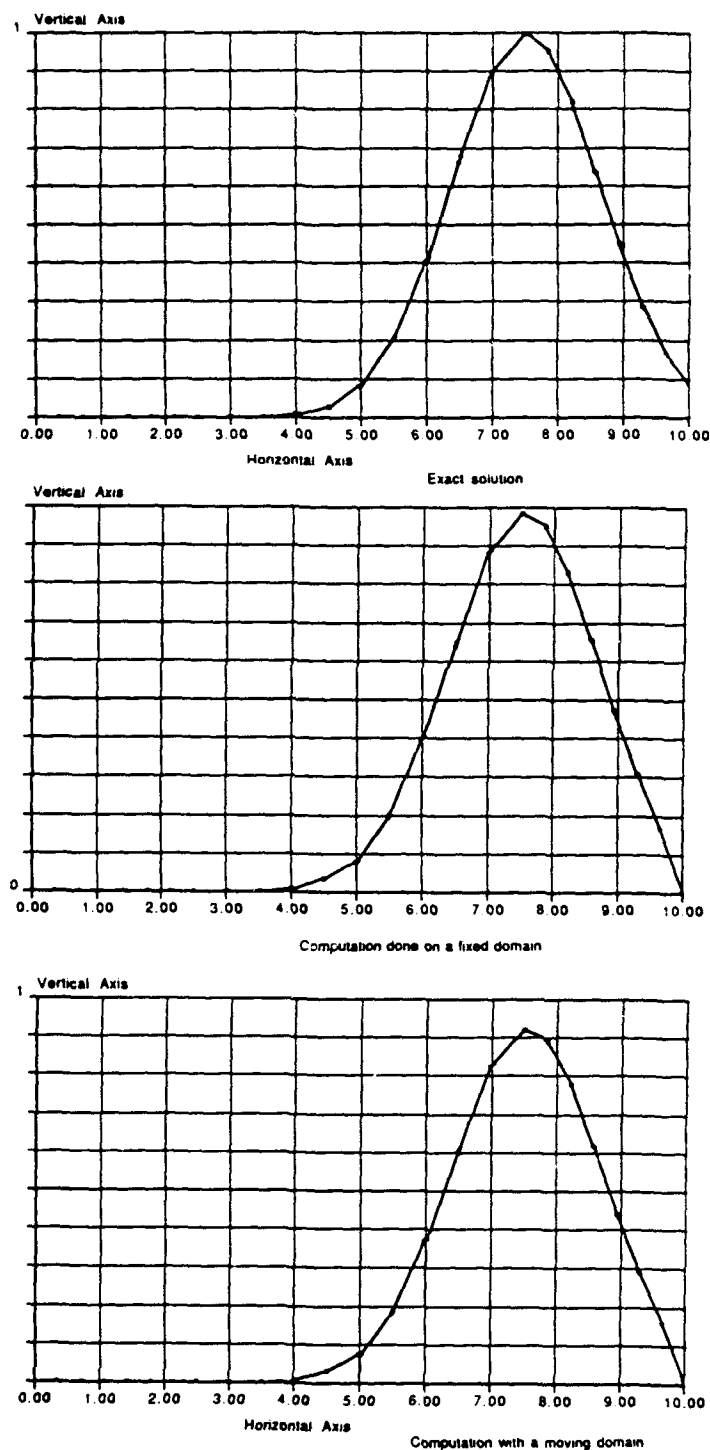


Fig. 2a. Central cross sections of the exponential hill with $\nu = 0$ obtained with the characteristic-Galerkin method after 14 time steps: exact solution (top), for fixed mesh (middle), and for rotating mesh (bottom).

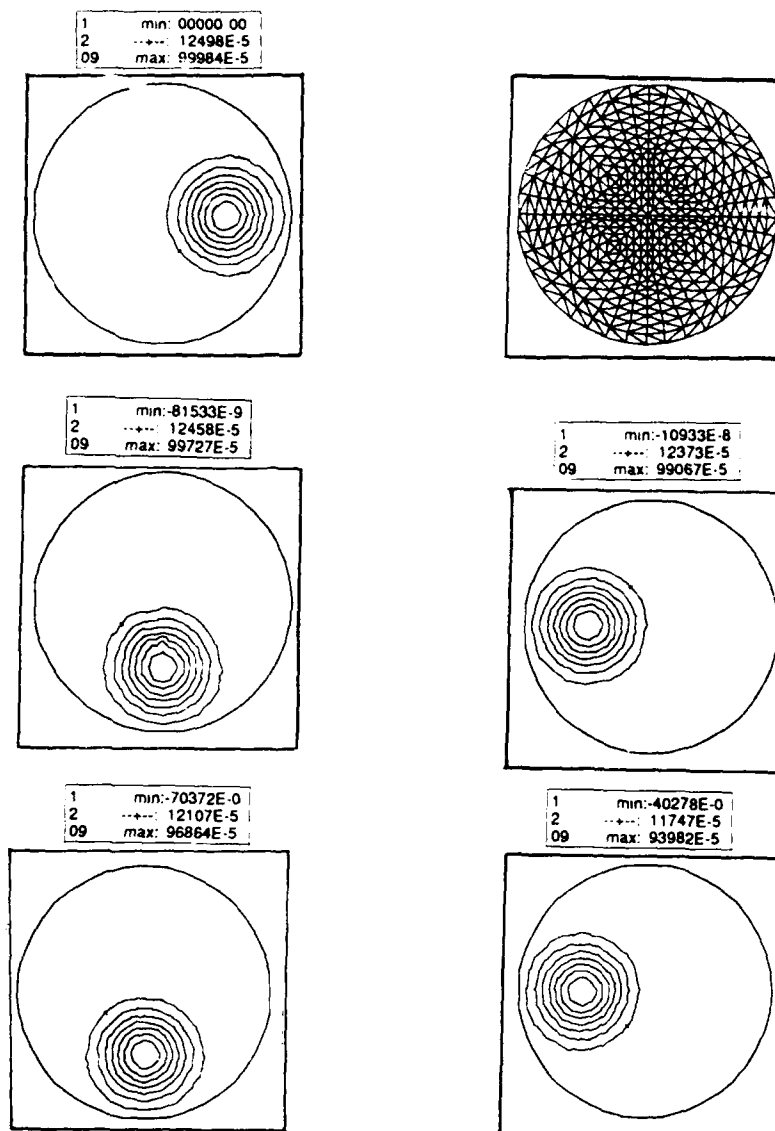


Fig. 2b. Contour plots of the exponential hill with $\nu = 0$ obtained with the characteristic-Galerkin method after 14 time steps: initial condition and mesh (top), for fixed mesh (middle), and for rotating mesh (bottom).

At each iteration in time, the vertices q' of the triangulation are moved by integrating the equation

$$\frac{dx}{dt} = v(x(t), t), \quad x(t^0) = q',$$

with ten steps of the forward Euler scheme.

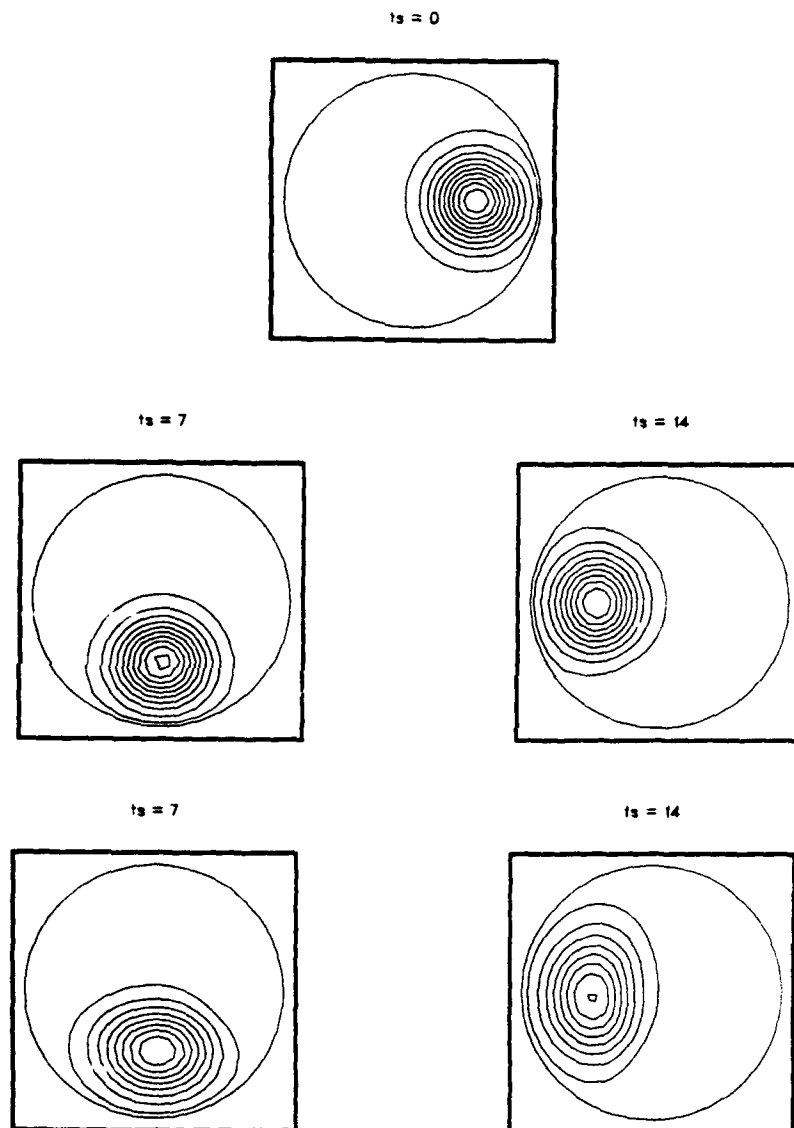


Fig. 2*. Contour plots of the exponential hill with $\nu = 0$ obtained with the least-square Galerkin method: initial condition (top), for fixed mesh (middle), and for rotating mesh (bottom).

We tested several cases in which values of the parameters in the definition of $\beta(x, t)$ are $\beta_0 = 0$ or -1 , $\beta_r = 0$ or 0.25 , $\alpha = 0$ or 1 and $\nu = 0$ or 0.01 . In all cases, the solutions obtained with the characteristic-Galerkin and the GLS/ST methods are presented. We use a time step size of $\pi/14$.

Rotating hill with fixed and rotating meshes. We set $\beta_0 = 0$ for the fixed mesh and $\beta_0 = -1$ for the rotating mesh. We computed in each case for 14 time steps. Figures 2a, 2b and 2* show the solutions for the case with $\nu = 0$ at time steps 7 and 14, while Figs. 3 and 3* show the solutions for the case with $\nu = 0.01$ at the same time steps.

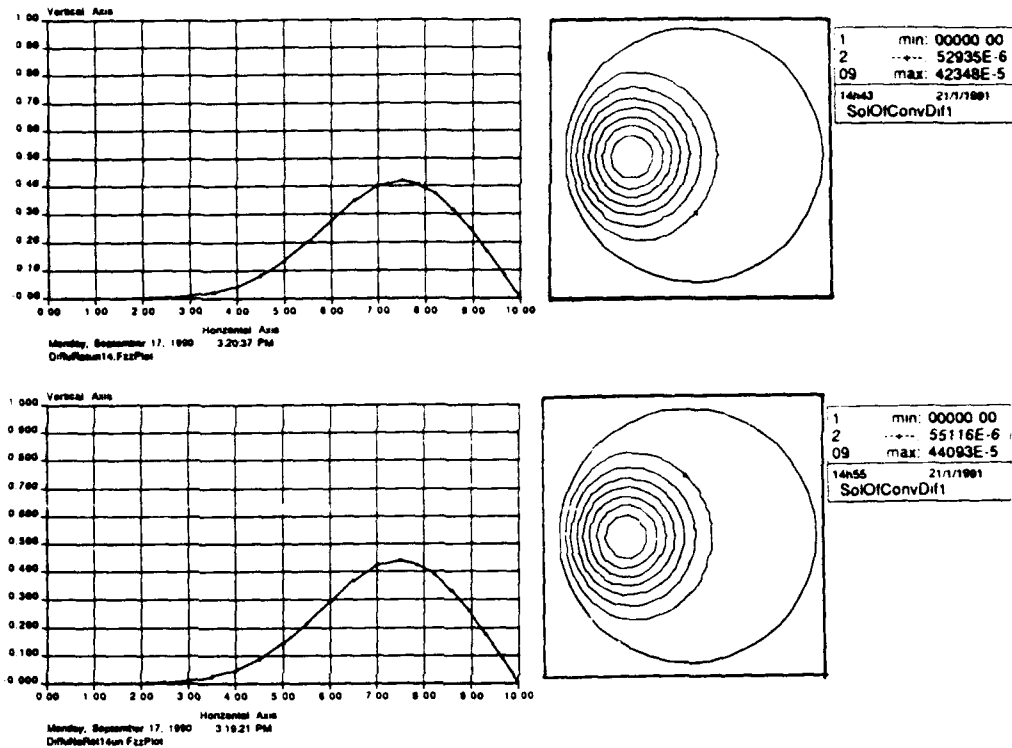


Fig. 3. Central cross sections and contour plots of the exponential hill with $\nu = 0.01$ obtained with the characteristic-Galerkin method after 14 time steps: for fixed mesh (top), and for rotating mesh (bottom).

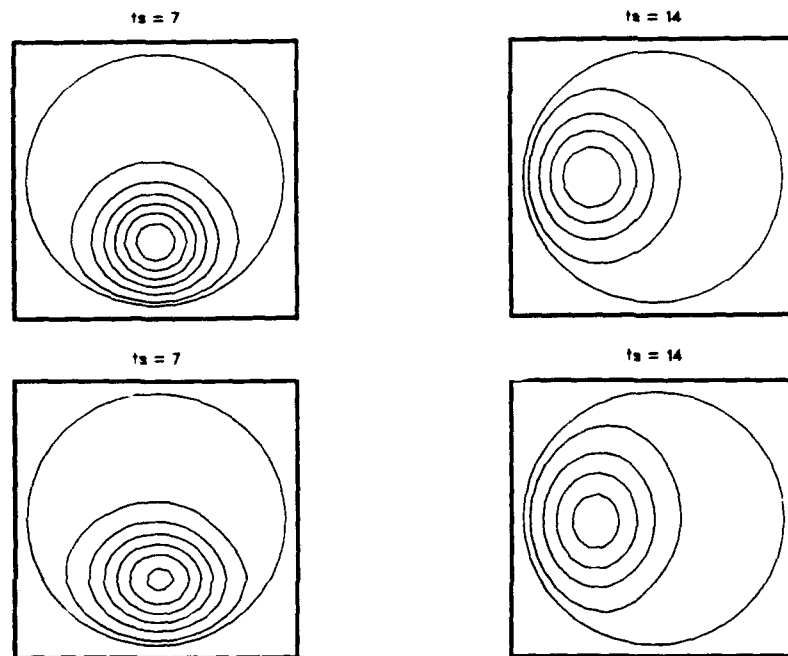


Fig. 3*. Contour plots of the exponential hill with $\nu = 0.01$ obtained with the least-square Galerkin method: for fixed mesh (top), and for rotating mesh (bottom).

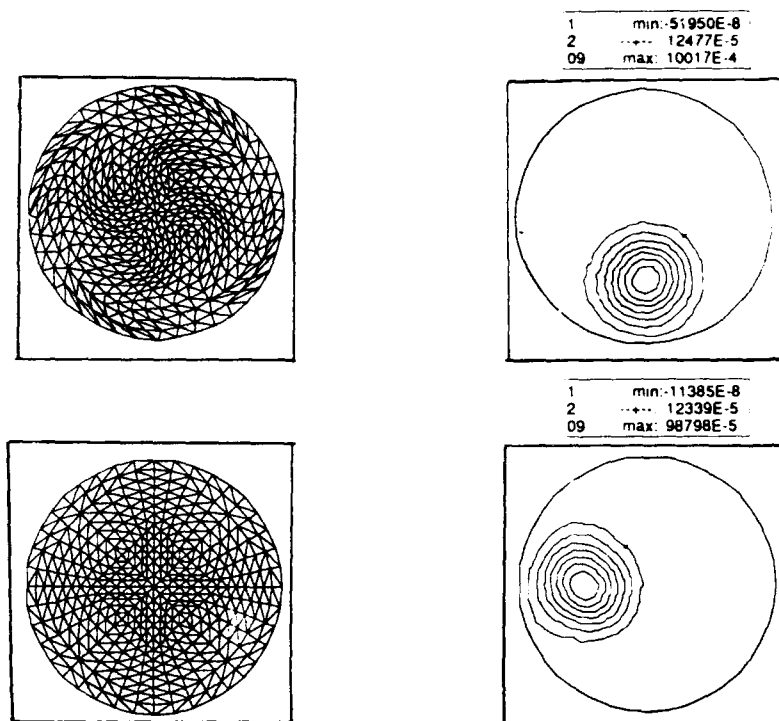


Fig. 4. Meshes and contour plots of the exponential hill with $\nu = 0$ for stretching mesh obtained with the characteristic-Galerkin method: after 7 time steps (top), and after 14 time steps (bottom).

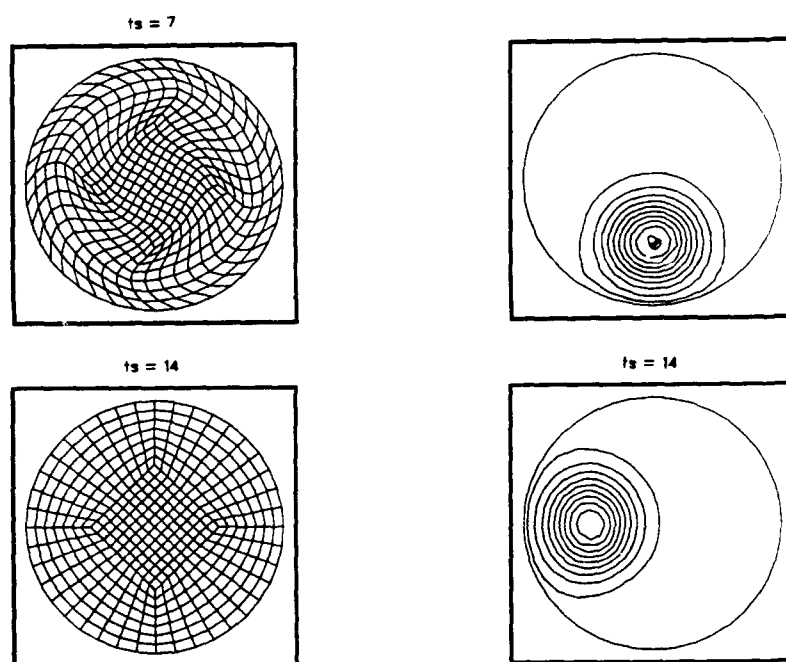


Fig. 4*. Meshes and contour plots of the exponential hill with $\nu = 0$ for stretching mesh obtained with the least-square Galerkin method: after 7 time steps (top), and after 14 time steps (bottom).

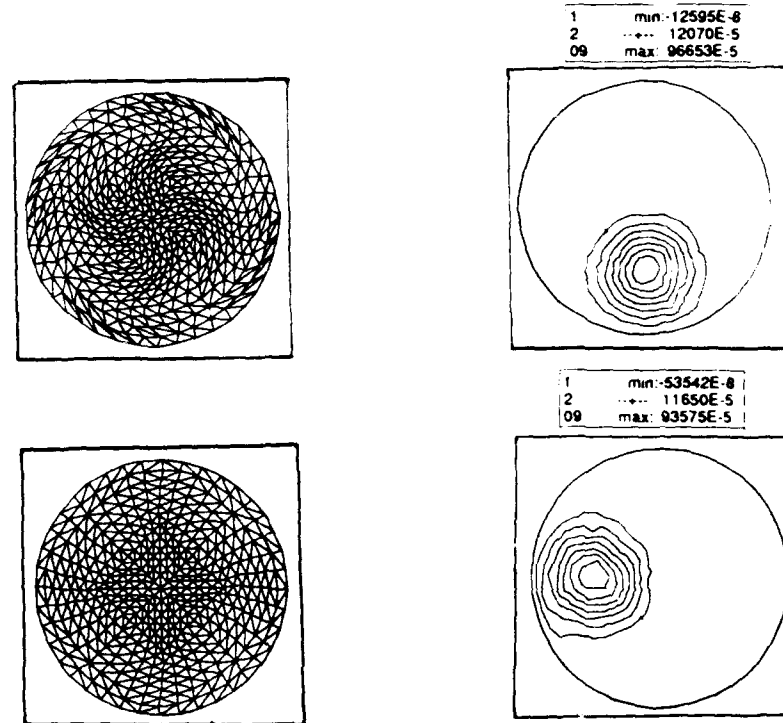


Fig. 5. Meshes and contour plots of the exponential hill with $\nu = 0$ for rotating and stretching mesh obtained with the least-square Galerkin method: after 7 time steps (top), and after 14 time steps (bottom).

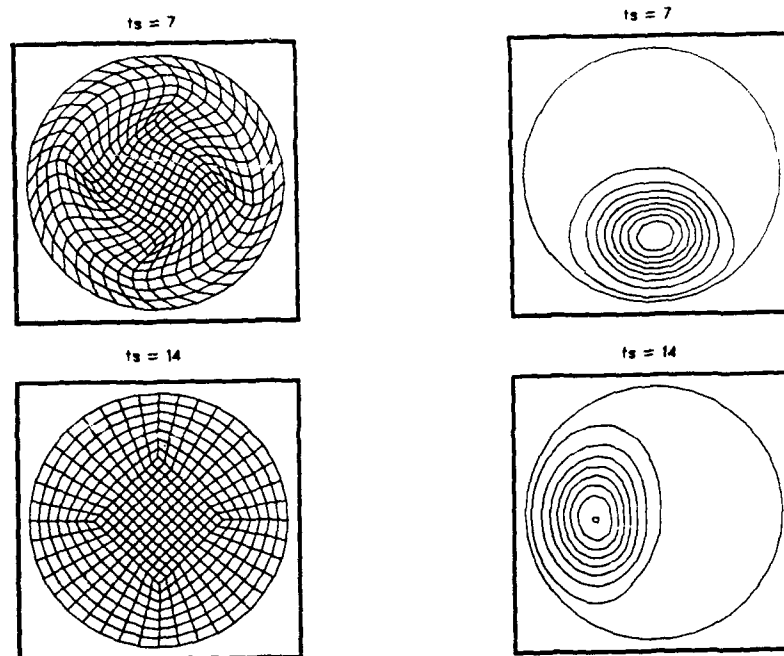


Fig. 5*. Meshes and contour plots of the exponential hill with $\nu = 0$ for rotating and stretching mesh obtained with the least-square Galerkin method: after 7 time steps (top), and after 14 time steps (bottom).

Rotating hill with stretching mesh. In this case, we set $\beta_r = 0.25$ to describe the motion of the mesh. The viscosity is set to 0. Figures 4 and 4* show the meshes and the solutions at time steps 7 and 14.

Rotating hill with rotating and stretching mesh. In this case, we set $\beta_r = -1$ and $\beta_s = 0.25$ to describe the motion of the mesh. The viscosity is set to 0. Figures 5 and 5* show the meshes and the solutions at time steps 7 and 14.

The computations for the characteristic-Galerkin were done on a MacIIfx. Each run takes approximately 30 min and 1 Mbyte. The computations for the GLS/ST were done on a Cray 2.

4. Conclusions

We have compared the characteristic-Galerkin and Galerkin/least-squares space-time formulations (which are both suitable for time-dependent domains) based on error estimates and numerical performance for a test problem governed by the advection-diffusion equation. We have shown that the formulation of the problem with either method is well-posed and the order of accuracy is the same as when the spatial domain is fixed, that is, $O(h + \delta t)$ for the characteristic-Galerkin formulation and $O(h^{3/2} + \delta t^{3/2})$ for the GLS/ST formulation. It was brought to our attention, at the galley proofs time, that related convergence studies appear in [20, 21]. The test problem involves the transport of an exponential hill in a rotational flow field. The results obtained with the two methods for various combinations of the Peclet number and mesh motion show that the numerical performance of these two methods are very good.

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